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A Multiplication Formula for the Modified Caldero–Chapoton Map

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Abstract

A frieze in the modern sense is a map from the set of objects of a triangulated category C to some ring. A frieze X is characterised by the property that if $\tau x \to y \to x$ is an Auslander–Reiten triangle in C, then $X(\tau x)X(x) - X(y) = 1$. The canonical example of a frieze is the (original) Caldero–Chapoton map, which send objects of cluster categories to elements of cluster algebras. Holm and Jørgensen (Nagoya Math J 218:101–124, 2015; Bull Sci Math 140:112–131, 2016), the notion of generalised friezes is introduced. A generalised frieze X' has the more general property that $X'(\tau x)X'(x) - X'(y) \in \{0, 1\}$. The canonical example of a generalised frieze is the modified Caldero–Chapoton map, also introduced in Holm and Jørgensen (2015, 2016). Here, we develop and add to the results in Holm and Jørgensen (2016). We define Condition F for two maps α and β in the modified Caldero–Chapoton map, and in the case when C is 2-Calabi–Yau, we show that it is sufficient to replace a more technical "frieze-like" condition from Holm and Jørgensen (2016). We also prove a multiplication formula for the modified Caldero–Chapoton map, which significantly simplifies its computation in practice.

Keywords Auslander–Reiten triangle · Categorification · Cluster algebra · Cluster category · Cluster tilting object · Cluster tilting subcategory · Rigid object · Rigid subcategory · Triangulated category

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1 Introduction

1.1 Summary

This paper focuses around two main topics: generalised friezes with integer values (see [14]) and generalised friezes taking values inside a Laurent polynomial ring (see [15]).

A frieze is a map $X : \text{obj}\,\mathsf{C} \to A$, where C is some triangulated category with Auslander–Reiten (AR) triangles and A is a ring, such that the following exponential conditions are satisfied:

$$X(0) = 1 \text{ and } X(a \oplus b) = X(a)X(b), \tag{1.1}$$

and if $\tau x \to y \to x$ is an AR triangle, then

$$X(\tau x)X(x) - X(y) = 1. \tag{1.2}$$

The canonical example of a frieze is the Caldero–Chapoton map, which we recall in Sect. 1.4.

Generalised friezes are similarly defined maps $X' : \text{obj}\,\mathsf{C} \to A$, also satisfying the exponential conditions in (1.1), however we permit the more general property that

$$X'(\tau x)X'(x) - X'(y) \in \{0, 1\}. \tag{1.3}$$

The canonical example of a generalised frieze is the modified Caldero–Chapoton map, which we recall in Sect. 1.6. The arithmetic version π, with integer values, is defined in Eq. (1.5), whilst the more general version ρ, taking values inside a Laurent polynomial ring, is defined in Eq. (1.8).

The modified Caldero–Chapoton map was introduced in [15], and we improve and add to the results of that paper. When working with a 2-Calabi–Yau category, we manage to replace the technical "frieze-like" condition (see [15, def. 1.4]) for the maps α and β in the generalised Caldero–Chapoton map (Eq. 1.8), by our so-called Condition F (see Definition 3.1). This condition significantly simplifies the frieze-like condition and demonstrates the roles of α and β. We will see that α plays the role of a "generalised index", whilst β provides a correction term to α being exponential over a distinguised triangle.

We use this to establish a multiplication formula for the modified Caldero–Chapoton map ρ (see Theorem 6.2), allowing its computation in practice. In [15], the computation of ρ is not addressed. However, our multiplication formula does address the computation, and does so in a simpler manner than merely applying the definition. In particular, the formula allows us to compute values of ρ without calculating Euler characteristics of submodule Grassmannians which are otherwise part of the definition of ρ.

1.2 Cluster Categories

Cluster categories were first introduced by Buan et al. [4] as a means of understanding the 'decorated quiver representations' introduced by Reineke et al. [16]. Let Q be a finite quiver with no loops or cycles, and consider the category $\mathsf{mod}\,\mathbb{C}Q$ of finitely generated modules over the path algebra $\mathbb{C}Q$. Then, set

$$\mathsf{D}(Q) = \mathsf{D}^b(\mathsf{mod}\,\mathbb{C}Q),$$

the bounded derived category of $\mathsf{mod}\,\mathbb{C}Q$.

The cluster category of type Q, denoted $\mathsf{C}(Q)$, is defined to be the orbit category of $\mathsf{D}(Q)$ under the action of the cyclic group generated by the autoequivalence $\tau^{-1}\Sigma = S^{-1}\Sigma^2$, where

τ is the Auslander–Reiten translation, Σ the suspension functor and S the Serre functor. That is,

$$\mathsf{C}(Q) = \mathsf{D}(Q)/(S^{-1}\Sigma^2).$$

The objects in $\mathsf{C}(Q)$ are the same as those in $\mathsf{D}(Q)$, however, the morphism sets in $\mathsf{C}(Q)$ are given by

$$\mathrm{Hom}_{\mathsf{C}(Q)}(X, Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(Q)}(X, (S^{-1}\Sigma^2)^n Y).$$

We note that $\mathsf{C}(Q)$ possesses a triangulated structure, it is $\mathbb{C}$-linear, Hom-finite, Krull-Schmidt and 2-Calabi–Yau, meaning that its Serre functor is Σ^2. It is also essentially small and has split idempotents. See [4, sec. 1]

The cluster category of Dynkin type A_n, denoted by $\mathsf{C}(A_n)$, has a very nice polygon model associated to it. This is due to Caldero et al. [7], who defined, for finite quivers of type A_n, an equivalent category to the cluster category in [4] in a totally different manner. This is done using a triangulation of a regular $(n+3)$-gon P, with objects and morphisms described in [7, sec. 2].

The category $\mathsf{C}(A_n)$ carries several nice properties. There is a bijection between the set of indecomposables of $\mathsf{C}(A_n)$, denoted indec $\mathsf{C}(A_n)$, and the set of diagonals of P. We also identify each edge of P with the zero object in $\mathsf{C}(A_n)$. Applying the suspension functor Σ to an indecomposable corresponds to rotating the endpoints of the associated diagonal one vertex clockwise. That is, if the vertices of P are labelled in an anticlockwise fashion with the set $\{1, 2, \ldots, n\}$, then for some indecomposable $\{i, j\}$, where i and j are vertices of P, we have

$$\Sigma\{i, j\} = \{i-1, j-1\}.$$

Such coordinates should clearly be taken modulo $n+3$.

Identifying indecomposables of $\mathsf{C}(A_n)$ with the diagonals of P carries the convenient property that for $a, b \in \mathrm{indec}\,\mathsf{C}(A_n)$,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathsf{C}(A_n)}(a, b) = \begin{cases} 1, & \text{if } a \text{ and } b \text{ cross} \\ 0, & \text{if } a \text{ and } b \text{ do not cross.} \end{cases}$$

1.3 The Auslander–Reiten Quiver

The Auslander–Reiten quiver for $\mathsf{C}(A_n)$ is $\mathbb{Z}A_n$ modulo a glide reflection. A coordinate system may be put on the quiver, matching up with the diagonals of the $(n+3)$-gon [see Fig. 1 for an example of $\mathsf{C}(A_5)$].

The Auslander–Reiten triangles in $\mathsf{C}(A_n)$ take the form

$$\{i-1, j-1\} \to \{i-1, j\} \oplus \{i, j-1\} \to \{i, j\},$$

where if $\{i-1, j\}$ or $\{i, j-1\}$ correspond to an edge of P, then they should be taken to be zero, see [5, lem. 3.15]. Notice that each AR triangle can be realised from a diamond in the AR quiver. That is, if

$$
\begin{tikzcd}
\{5,7\} \arrow[dr] & & \{6,8\} \arrow[dr] & & \{1,7\} \arrow[dr] & & \{2,8\} \arrow[dr] & & \{1,3\} \arrow[dd, dotted] \\
& \{5,8\} \arrow[ur] \arrow[dr] & & \{1,6\} \arrow[ur] \arrow[dr] & & \{2,7\} \arrow[ur] \arrow[dr] & & \{3,8\} \arrow[ur] \arrow[dr] & \\
\{4,8\} \arrow[uu, dotted] \arrow[ur] \arrow[dr] & & \{1,5\} \arrow[ur] \arrow[dr] & & \{2,6\} \arrow[ur] \arrow[dr] & & \{3,7\} \arrow[ur] \arrow[dr] & & \{4,8\} \arrow[dd, dotted] \\
& \{1,4\} \arrow[ur] \arrow[dr] & & \{2,5\} \arrow[ur] \arrow[dr] & & \{3,6\} \arrow[ur] \arrow[dr] & & \{4,7\} \arrow[ur] \arrow[dr] & \\
\{1,3\} \arrow[uu, dotted] \arrow[ur] & & \{2,4\} \arrow[ur] & & \{3,5\} \arrow[ur] & & \{4,6\} \arrow[ur] & & \{5,7\}
\end{tikzcd}
$$

Fig. 1 The Auslander–Reiten quiver for $\mathsf{C}(A_5)$. The dotted lines are identified with opposite orientations

$$
\begin{tikzcd}
& b \arrow[dr] & \\
a \arrow[ur] \arrow[dr] & & d \\
& c \arrow[ur] &
\end{tikzcd}
$$

is a diamond inside the AR quiver, then

$$a \to b \oplus c \to d,$$

is an AR triangle. If a and d sit on the upper boundary of the AR quiver, then b should be taken as zero, whereas if a and d sit on the lower boundary, then c is taken to be zero. Note that a frieze X on $\mathsf{C}(A_n)$ satisfies:

$$X(a)X(d) - X(b)X(c) = 1. \tag{1.4}$$

1.4 The Caldero–Chapoton Map

The (original) Caldero–Chapoton map is a map which sends certain (so-called reachable) indecomposable objects of a cluster category to cluster variables of the corresponding cluster algebra, see [8, sec. 4.1]. The map, which we denote by γ, depends on a cluster tilting object T inside the cluster category and makes precise the idea that the cluster category is a categorification of the cluster algebra. It is required that the category is 2-Calabi–Yau (for example a cluster category), and it is a well known property of γ that it is a frieze (see [1, def. 1.1], [6, prop. 3.10], [12, thm.]). That is, the Caldero–Chapoton map is a map $\gamma\ : \operatorname{obj} \mathsf{C} \to A$, where A is a certain ring, which satisfies the property in Eq. (1.2), as well as the exponential conditions in (1.1).

1.5 Frieze Patterns

Frieze patterns were first introduced by Conway and Coxeter [9,10]. An example of such a frieze pattern, known as a Conway–Coxeter frieze, is given in Fig. 2. This frieze pattern is obtained from the original Caldero–Chapoton map γ by omitting the arrows from the AR quiver of $\mathsf{C}(A_7)$ and replacing each vertex by the value of γ applied to that indecomposable.

In formal terms, for some positive integer n, a frieze pattern is an array of n offset rows of positive integers. Each diamond

$$
\begin{array}{ccccccccccc}
4 & & 4 & & 1 & & 2 & & 2 & & 4 \\
& 15 & & 3 & & 1 & & 3 & & 7 & \\
11 & & 11 & & 2 & & 1 & & 10 & & 5 \\
& 8 & & 7 & & 2 & & 3 & & 7 & \\
5 & & 5 & & 3 & & 2 & & 2 & & 11 \\
& 3 & & 2 & & 5 & & 1 & & 3 & \\
4 & & 1 & & 3 & & 2 & & 1 & & 4
\end{array}
$$

Fig. 2 A frieze on the cluster category of Dynkin type A_7. The dotted lines are identified with opposite orientations

$$
\begin{array}{ccc}
& \alpha & \\
\beta & & \eta \\
& \delta &
\end{array}
$$

in a frieze pattern satisfies a so-called determinant property in that

$$\beta\eta - \alpha\delta = 1.$$

If β and η are on the top row of the frieze, then the determinant property becomes

$$\beta\eta - \delta = 1,$$

and if β and η are on the bottom row, the determinant property becomes

$$\beta\eta - \alpha = 1.$$

Observe that the Caldero–Chapoton map satisfies these equations by virtue of Eq. (1.4), that is, because it is a Conway–Coxeter frieze. Conway–Coxeter frieze patterns are known to be invariant under a glide reflection. In this case, a region of the frieze pattern, known as a fundamental domain, is enough to produce the whole frieze pattern by repeatedly performing a glide reflection.

1.6 A Modified Caldero–Chapoton Map

We assume in the rest of the paper that C is an essentially small, $\mathbb{C}$-linear, Hom-finite, triangulated category, which is Krull–Schmidt and has AR triangles. Holm and Jørgensen [14] introduce a modified version of the Caldero–Chapoton map, which we denote by π, that relies on a rigid object $R \in \operatorname{obj} \mathsf{C}$, a much weaker condition than that of being a cluster tilting object. We say that an object R is rigid if

$$\operatorname{Hom}_{\mathsf{C}}(R, \Sigma R) = 0.$$

We also note that π does not require that the category is 2-Calabi–Yau, allowing us to work with a category C that is more general than a cluster category.

Consider the endomorphism ring $E = \mathrm{End}_{\mathsf{C}}(R)$, and define $\operatorname{mod} E$ to be the category of finite dimensional right E-modules. Then, there is a functor

$$G : \mathsf{C} \to \operatorname{mod} E$$
$$c \mapsto \mathrm{Hom}_{\mathsf{C}}(R, \Sigma c).$$

For some object $c \in \mathsf{C}$, the modified Caldero–Chapoton map is then defined by the formula:

$$\pi(c) = \chi(\mathrm{Gr}(Gc)), \tag{1.5}$$

where Gr denotes the Grassmannian of submodules and χ is the Euler characteristic defined by cohomology with compact support (see [13, p. 93]).

It is proved in [14] that π is a generalised frieze; that is, as well as the exponential properties given in Eq. (1.1) it satisfies the property given in Eq. (1.3).

Define $\mathsf{R} = \operatorname{add} R$, the full subcategory whose objects are finite direct sums of the summands of R (see Sect. 2 for details of this setup). This full subcategory, which is clearly closed under direct sums and summands, is rigid in the sense that $\mathrm{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma\mathsf{R}) = 0$. A multiplication formula for computing π is also proved in [14]. Let $m \in \operatorname{indec} \mathsf{C}$ and $r \in \operatorname{indec} \mathsf{R}$ satisfy that $\mathrm{Ext}^1_{\mathsf{C}}(m, r)$ and $\mathrm{Ext}^1_{\mathsf{C}}(r, m)$ both have dimension one over $\mathbb{C}$. Then, there are nonsplit triangles

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \ , \ r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r, \tag{1.6}$$

that are unique up to isomorphism. It is proved in [14] that

$$\pi(m) = \pi(a) + \pi(b). \tag{1.7}$$

This formula can be applied iteratively to compute values of π.

Holm and Jørgensen [15] redefine the modified Caldero–Chapoton map in a more general manner (the work in [14] is a special case of that in [15]). They define ρ by

$$\rho(c) = \alpha(c) \sum_e \chi(\mathrm{Gr}_e(Gc))\beta(e). \tag{1.8}$$

Here the sum is taken over $e \in \mathrm{K}_0(\operatorname{mod} E)$, the Grothendieck group of the abelian category $\operatorname{mod} E$, and $\mathrm{Gr}_e(Gc)$ is the Grassmannian of E-submodules $M \subseteq Gc$ with K_0-class satisfying $[M] = e$. The maps

$$\alpha : \operatorname{obj} \mathsf{C} \to A \text{ and } \beta : \mathrm{K}_0(\operatorname{mod} E) \to A \tag{1.9}$$

are both exponential maps in the sense that

$$\begin{aligned} \alpha(0) = 1 \ , \ \alpha(x \oplus y) = \alpha(x)\alpha(y), \\ \beta(0) = 1 \ , \ \beta(e + f) = \beta(e)\beta(f), \end{aligned} \tag{1.10}$$

and A is some commutative ring, see [15, setup 1.1].

When the maps α and β satisfy a technical "frieze-like" condition, given in [15, def. 1.4], the map ρ becomes a generalised frieze, as in Eq. (1.3).

1.7 This Paper

In this paper, we show a simpler condition on α and β than that in [15], which implies that ρ is a generalised frieze. We also show that a similar multiplication formula to that in Eq. (1.7) holds with ρ instead of π. This permits a simpler iterative procedure for computing ρ than the one given in [15].

We give the following definition:

Definition 3.1 (*Condition F*) We say that the maps α and β satisfy Condition F if for each triangle

$$x \xrightarrow{\varphi} y \xrightarrow{\omega} z \xrightarrow{\psi} \Sigma x$$

in C such that Gx, Gy and Gz have finite length in $\mathsf{Mod}E$, the following property holds:

$$\alpha(y) = \alpha(x \oplus z)\beta([\operatorname{Ker} G\varphi]).$$

The following main result shows that when C is 2-Calabi–Yau, this definition is sufficient to replace the frieze-like condition from [15, def. 1.4].

Theorem 3.2 *Assume that the exponential maps $\alpha : \operatorname{obj}\mathsf{C} \to A$ and $\beta : K_0(\mathsf{mod}E) \to A$ from Eq. (1.9) satisfy Condition F from Definition 3.1. Then, the modified Caldero–Chapoton map ρ of Eq. (1.8) is a generalised frieze in the sense that it satisfies Eq. (1.3), as well as the exponential conditions in (1.1).*

We then proceed by proving in Lemma 4.1 that the construction of α and β in [15, def. 2.8] satisfies Condition F. Then, Theorem 3.2, together with Lemma 4.1, recovers a main result of [15], proving that the construction of α and β in [15, def. 2.8] results in ρ being a generalised frieze.

In Sect. 6, we prove the multiplication formula for ρ, similar to the arithmetic case for π in (1.7). This multiplication formula is as follows:

Theorem 6.2 *Let $m \in \operatorname{indec}\mathsf{C}$ and $r \in \operatorname{indec}\mathsf{R}$ such that $\operatorname{Ext}^1_{\mathsf{C}}(r, m)$ and $\operatorname{Ext}^1_{\mathsf{C}}(m, r)$ both have dimension one over $\mathbb{C}$. Then, there are nonsplit triangles*

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \qquad \text{and} \qquad r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r,$$

with δ and ζ nonzero. Let Gm have finite length in $\mathsf{Mod}E$, then

$$\rho(r)\rho(m) = \rho(a) + \rho(b).$$

This theorem can be applied inductively in order to simplify the computation of values of ρ. For some indecomposable m in C, one may find an a and b (which are the middle terms of the nonsplit extensions), such that $\rho(m) = \rho(a) + \rho(b)$. The theorem can then be reapplied to find each of $\rho(a)$ and $\rho(b)$. The process will eventually terminate at the stage where calculating ρ of some indecomposable reduces to calculating the index of that indecomposable. Substituting back into the equation allows a simple calculation of $\rho(m)$.

We illustrate the procedure in Sect. 7 by computing ρ of an indecomposable in the Auslander–Reiten quiver for $\mathsf{C}(A_5)$. This retrieves one of the vertices of the AR quiver in Fig. 3. Note that this example already appeared in [15, sec. 3], but Theorem 6.2 makes our computation much simpler.

This paper is organised as follows. Section 2 gives an essential background to the modified Caldero–Chapoton map, and explains some important results from [15]. Section 3 introduces Condition F and proves Theorem 3.2, whilst Sect. 4 shows how one can construct maps α and β satisfying this condition. Section 5 demonstrates how to find the multiplication formula

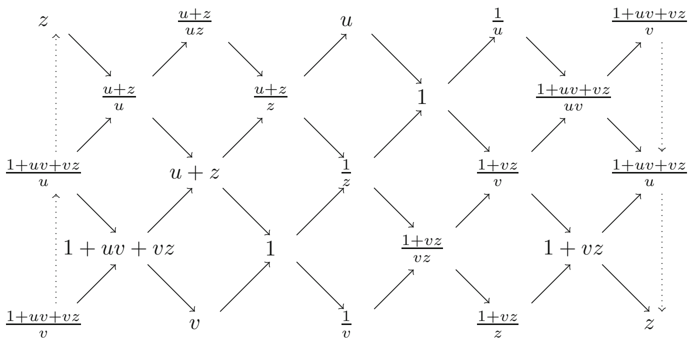

Fig. 3 The Auslander–Reiten quiver of the cluster category of Dynkin type A_5. The vertices have been replaced with values of the modified Caldero–Chapoton map ρ. Again, the dotted lines are identified with oppostite orientations

for π, Sect. 6 adapts this formula to ρ by proving the multiplication formula in Theorem 6.2, and Sect. 7 shows why this formula is useful.

It should be noted that Sects. 2 and 5 contain no original work, however they provide an essential setup for subsequent sections in the paper.

2 A Modified Caldero–Chapoton Map: A Functorial Viewpoint

In this section, we redefine the modified Caldero–Chapoton map in detail, using an equivalent, functorial viewpoint, allowing us simpler calculations throughout the rest of the paper. To set up, we will follow the construction in [15, sec. 2]. We add the assumption that C is 2-Calabi–Yau. We let R be a functorially finite subcategory of C, which is closed under sums and summands and rigid; that is, $\mathrm{Hom}_{\mathsf{C}}(\mathsf{R}, \Sigma\mathsf{R}) = 0$.

We also assume that C has a cluster tilting subcategory T, belonging to a cluster structure in the sense of [3, sec. II.1]. We additionally require that $\mathsf{R} \subseteq \mathsf{T}$. Note that the Auslander–Reiten translation for C is

$$\tau = \Sigma,$$

and its Serre functor is $S = \Sigma^2$.

There is a functor G, defined by:

$$\begin{aligned} G : \mathsf{C} &\to \mathsf{Mod}\,\mathsf{R} \\ c &\mapsto \mathrm{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{R}} \end{aligned} \tag{2.1}$$

where $\mathsf{Mod}\,\mathsf{R}$ denotes the category of $\mathbb{C}$-linear contravariant functors $\mathsf{R} \to \mathsf{Vect}\,\mathbb{C}$. It is a $\mathbb{C}$-linear abelian category, and we denote by $\mathsf{fl}\,\mathsf{R}$ the full subcategory of $\mathsf{Mod}\,\mathsf{R}$, consisting of all finite length objects.

The modified Caldero–Chapoton map is then defined as in Eq. (1.8), where the sum is now taken over $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$. Here, we have

$$\alpha : \mathrm{obj}\,\mathsf{C} \to A\ ,\ \ \beta : \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \to A, \tag{2.2}$$

and these maps satisfy the exponential condition in Eq. (1.10)

For two objects $a, b \in \mathsf{C}$ such that Ga and Gb have finite length, it is known by [15, prop. 1.3] that ρ is also exponential; that is,

$$\rho(0) = 1\ ,\ \rho(a \oplus b) = \rho(a)\rho(b).$$

It therefore suffices to calculate ρ for each indecomposable object in C. Note that the formula for ρ only makes sense when Gc has finite length in $\operatorname{Mod} \mathsf{R}$. That is, we require that $Gc \in \operatorname{fl} \mathsf{R}$.

3 Condition F on the Maps α and β

We continue under the setup of Sect. 2. Consider the exponential maps α and β introduced earlier in Eq. (2.2). The following definition describes a canonical condition on α and β, which will later be used in Sect. 6 to prove a multiplication formula for ρ.

Definition 3.1 (*Condition F*) We say that the maps α and β satisfy Condition F if, for each triangle

$$x \xrightarrow{\varphi} y \xrightarrow{\omega} z \xrightarrow{\psi} \Sigma x \tag{3.1}$$

in C, such that Gx, Gy and Gz have finite length in $\operatorname{Mod} \mathsf{R}$, the following property holds:

$$\alpha(y) = \alpha(x \oplus z)\beta([\operatorname{Ker} G\varphi]).$$

We now prove a theorem showing that in the case when C is 2-Calabi–Yau, Condition F can replace the definition of the frieze-like condition defined in [15, def. 1.4].

Theorem 3.2 *Assume that the exponential maps* $\alpha : \operatorname{obj} \mathsf{C} \to A$ *and* $\beta : K_0(\operatorname{fl} \mathsf{R}) \to A$ *from Eq.* (2.2) *satisfy Condition F from Definition* 3.1*. Then, the modified Caldero–Chapoton map* ρ*, defined in Eq.* (1.8)*, is a generalised frieze in the sense that, where defined, it satisfies Eq.* (1.3)*, as well as the exponential conditions in* (1.1).

Proof Note that by [15, thm. 1.6], if

$$\Delta = \left(\Sigma c \xrightarrow{\xi} b \longrightarrow c\right)$$

is an AR triangle in C such that Gc and $G\Sigma c$ have finite length in $\operatorname{Mod} \mathsf{R}$, and α and β are frieze-like maps for Δ in the sense of [15, def. 1.4], then $\rho(\Sigma c)\rho(c) - \rho(b) \in \{0, 1\}$. Hence, it suffices to show that α and β are frieze-like for each Δ. We first note that if $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$, then $G(\Delta)$ is a short exact sequence, see [14, lem. 1.12(iii)]. We now check the three cases of [15, def. 1.4].

Case (i) Assume $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$ and $G(\Delta)$ is a split short exact sequence. That is,

$$0 \longrightarrow G(\Sigma c) \xrightarrow{G\xi} Gb \longrightarrow Gc \longrightarrow 0,$$

is split short exact. It follows immediately that $G\xi$ has trivial kernel. Applying Condition F to Δ, we obtain:

$$\begin{aligned}\alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\xi]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c),\end{aligned}$$

where the final = is due to β being exponential.

Case (ii) first part Assume $c \notin \mathsf{R} \cup \Sigma^{-1}\mathsf{R}$ and $G(\Delta)$ is a nonsplit short exact sequence. Then, by the same working as in Case (i), we see that:

$$\alpha(b) = \alpha(c \oplus \Sigma c).$$

Now, consider the following triangle in C:

$$c \xrightarrow{\nu} 0 \longrightarrow \Sigma c \xrightarrow{1_{\Sigma c}} \Sigma c.$$

Applying Condition F and using the fact that α is exponential, we obtain:

$$\begin{aligned} 1 &= \alpha(0) \\ &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\nu]) \\ &= \alpha(c \oplus \Sigma c)\beta([Gc]). \end{aligned}$$

We note that this manipulation works for any $c \in \operatorname{obj} \mathsf{C}$.

Case (ii) second part Let $c = \Sigma^{-1}r \in \Sigma^{-1}\mathsf{R}$. We showed in the first part of Case (ii) that

$$\alpha(c \oplus \Sigma c)\beta([Gc]) = 1.$$

Now, by [14, lem. 1.12(i)], $G(\Delta)$ becomes:

$$G(\Delta) = 0 \to \operatorname{rad} P_r \to P_r,$$

where P_r is the indecomposable projective $\operatorname{Hom}_{\mathsf{R}}(-, r)$ in $\operatorname{Mod} \mathsf{R}$, described in [14, sec. 1.5]. Therefore, $G\xi$ has zero kernel, and applyng Condition F shows:

$$\begin{aligned} \alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\xi]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c). \end{aligned}$$

Case (iii) Let $c = r \in \mathsf{R}$ and again consider the triangle

$$c \xrightarrow{\nu} 0 \longrightarrow \Sigma c \xrightarrow{1_{\Sigma c}} \Sigma c,$$

in C. Applying Condition F, we see

$$\begin{aligned} 1 &= \alpha(0) \\ &= \alpha(c \oplus \Sigma c)\beta([\operatorname{Ker} G\nu]) \\ &= \alpha(c \oplus \Sigma c)\beta(0) \\ &= \alpha(c \oplus \Sigma c), \end{aligned}$$

where the third = is since $G(c) = G(r) = 0$ and hence $\operatorname{Ker} G\nu = 0$.

Now, by [14, lem. 1.12(ii)], applying G to Δ gives the exact sequence:

$$I_r \xrightarrow{G\xi} \operatorname{corad} I_r \longrightarrow 0,$$

where I_r is the indecomposable injective $\operatorname{Hom}_{\mathsf{R}}(-, \Sigma^2 r)$ in $\operatorname{Mod} \mathsf{R}$, described in [14, sec. 1.10] and $\operatorname{corad} I_r$ is its coradical. Additionally, by [14, sec. 1.10], we know that there is the following short exact sequence:

$$0 \to S_r \to I_r \to \operatorname{corad} I_r \to 0,$$

where S_r in $\mathsf{Mod}\,\mathsf{R}$ is the simple object supported at r, see [2, prop. 2.2]. It follows that corad $I_r \cong I_r/S_r$ and we see that $\mathrm{Ker}\, G\xi = S_r$. So, by applying Condition F once again, we see that

$$\begin{aligned}\alpha(b) &= \alpha(c \oplus \Sigma c)\beta([\mathrm{Ker}\, G\xi])\\ &= \beta([\mathrm{Ker}\, G\xi])\\ &= \beta([S_r]),\end{aligned}$$

where the second = is since $\alpha(c \oplus \Sigma c) = 1$. □

4 Constructing Maps that Satisfy Condition F

We again continue under the setup of Sect. 2. We will show that there exist maps α and β satisfying Condition F, namely those given in [15, def. 2.8]. Let us first look at the necessary constructions behind the definitions of α and β in [15, def. 2.8].

Recall that T is some cluster tilting subcategory of C with $\mathsf{R} \subseteq \mathsf{T}$, and denote by $\mathrm{indec}\,\mathsf{T}$ the set of indecomposable objects in T. For each $t \in \mathrm{indec}\,\mathsf{T}$, one may find a unique indecomposable $t^* \in \mathrm{indec}\,\mathsf{C}$, called the mutation of t, such that replacing t with t^* gives rise to another cluster tilting subcategory T^*, see [3, sec. II.1]. Each such t and t^* fit into two exchange triangles (see [3, sec. II.1]):

$$t^* \to a \to t\ ,\ t \to a' \to t^*, \tag{4.1}$$

where $a, a' \in \mathrm{add}\big((\mathrm{indec}\,\mathsf{T}) \setminus t\big)$.

We denote by $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ the split Grothendieck group of the additive category T which has a basis formed by the set of indecomposables in T. We note that $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$ carries the relations that $[a \oplus b] = [a] + [b]$, where $[a]$ is used to denote the $\mathrm{K}_0^{\mathrm{split}}$-class of the object a in T.

Define S to be the full subcategory of C which is closed under direct sums and summands and has

$$\mathrm{indec}\,\mathsf{S} = \mathrm{indec}\,\mathsf{T} \setminus \mathrm{indec}\,\mathsf{R}, \tag{4.2}$$

and then consider the subgroup

$$N = \left\langle [a] - [a'] \;\middle|\; \begin{array}{l} s^* \to a \to s\ ,\ s \to a' \to s^* \text{ are exchange} \\ \text{triangles with } s \in \mathrm{indec}\,\mathsf{S} \end{array} \right\rangle \tag{4.3}$$

of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, defined in [15, def. 2.4]. Then, we denote by Q the canonical surjection

$$Q\ :\ \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \twoheadrightarrow \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N\ ,\ Q([t]) = [t] + N.$$

Now, for each $c \in \mathrm{obj}\,\mathsf{C}$, we may construct the element $\mathrm{ind}_{\mathsf{T}}(c) \in \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$, called the index of c with respect to the cluster tilting subcategory T. There exists a triangle $t' \to t \to c$ with $t, t' \in \mathsf{T}$ (see [11, sec. 1]). The index is then defined as:

$$\mathrm{ind}_{\mathsf{T}}(c) = [t] - [t'],$$

and is a well defined element of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})$.

Before giving the definition of α and β from [15, def. 2.8], it remains to recall how to construct the homomorphism θ from [15, sec. 2.6]. We do this by following the constructions in [15, sec. 2.5]. Since $\mathsf{R} \subseteq \mathsf{T}$, the inclusion functor $i : \mathsf{R} \hookrightarrow \mathsf{T}$ induces the exact functor

$$i^* : \mathsf{Mod}\,\mathsf{T} \to \mathsf{Mod}\,\mathsf{R},\ i^*(F) = F|_{\mathsf{R}},$$

where $\mathsf{Mod}\,\mathsf{T}$ is the abelian category of $\mathbb{C}$-linear contravariant functors $\mathsf{T} \to \mathsf{Vect}\,\mathbb{C}$. Now, by [2, prop. 2.3(b)], for each $t \in \operatorname{indec}\mathsf{T}$, there is a simple object $\overline{S}_t \in \mathsf{Mod}\,\mathsf{T}$ supported at t, and one may see that

$$i^*\overline{S}_t = \begin{cases} S_t & \text{if } t \in \operatorname{indec}\mathsf{R}, \\ 0 & \text{if } t \in \operatorname{indec}\mathsf{S}, \end{cases}$$

where S_t denotes the simple object in $\mathsf{Mod}\,\mathsf{R}$ supported at t. Due to i^* being exact, we can restrict it to the subcategories $\mathsf{fl}\,\mathsf{T}$ and $\mathsf{fl}\,\mathsf{R}$, made up of the finite length objects in $\mathsf{Mod}\,\mathsf{T}$ and $\mathsf{Mod}\,\mathsf{R}$, respectively. Then, there is an induced (surjective) group homomorphism

$$\kappa : \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \twoheadrightarrow \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}),$$

with the obvious property that

$$\kappa([\overline{S}_t]) = \begin{cases} [S_t] & \text{if } t \in \operatorname{indec}\mathsf{R}, \\ 0 & \text{if } t \in \operatorname{indec}\mathsf{S}. \end{cases}$$

For the category $\mathsf{Mod}\,\mathsf{T}$, there is a functor $\overline{G}$ similar to G from Eq. (2.1). It is defined by:

$$\begin{aligned} \overline{G} : \mathsf{C} &\to \mathsf{Mod}\,\mathsf{T} \\ c &\mapsto \operatorname{Hom}_{\mathsf{C}}(-, \Sigma c)|_{\mathsf{T}}. \end{aligned}$$

It is not hard to see that $\overline{G}$ has the property that $i^*\overline{G} = G$.

We define θ to be the group homomorphism making the following diagram commute:

$$\begin{tikzcd} \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \arrow[r, "\bar{\theta}"] \arrow[d, "\kappa"] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T}) \arrow[d, "Q"] \\ \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \arrow[r, "\theta"'] & \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N, \end{tikzcd} \tag{4.4}$$

where

$$\bar{\theta} : \mathrm{K}_0(\mathsf{fl}\,\mathsf{T}) \to \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})\ ,\ \bar{\theta}([\overline{S}_t]) = [a] - [a'],$$

where $a, a' \in \operatorname{add}\big((\operatorname{indec}\mathsf{T}) \setminus t\big)$ are from the exchange triangles for t in (4.1).

Now, we may recall from [15, def. 2.8] that the maps α: obj $\mathsf{C} \to A$ and $\beta : \mathrm{K}_0(\mathsf{fl}\,\mathsf{R}) \to A$ to a suitable ring A can be defined by:

$$\alpha(c) = \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(c)) \ \text{ and } \ \beta(e) = \varepsilon\theta(e), \tag{4.5}$$

where $\varepsilon : \mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N \to A$ is a suitably chosen exponential map, meaning that:

$$\varepsilon(0) = 1\ ,\ \varepsilon(a + b) = \varepsilon(a)\varepsilon(b). \tag{4.6}$$

Here a and b denote two elements of $\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N$.

Lemma 4.1 *The maps α and β from* (4.5) *satisfy Condition F.*

Proof Consider the following triangle

$$x \xrightarrow{\varphi} y \xrightarrow{\omega} z \xrightarrow{\psi} \Sigma x \tag{4.7}$$

from (3.1). Then, by definition, we have

$$\begin{aligned}\alpha(y) &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(y))\\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(x) + \mathrm{ind}_{\mathsf{T}}(z) - \mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C))\\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + \mathrm{ind}_{\mathsf{T}}(z) - \mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C) + N)\\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon(-\mathrm{ind}_{\mathsf{T}}(C) - \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C) + N)\\ &= (*),\end{aligned}$$

where C in C is some lifting of $\mathrm{Coker}\,\overline{G}(\Sigma^{-1}\omega)$ in the sense that $\overline{G}\Sigma^{-1}C = \mathrm{Coker}\,\overline{G}(\Sigma^{-1}\omega)$. In the above manipulation, the second = is due to [17, prop. 2.2] and the penultimate = occurs since ε is exponential, see Eq. (4.6).

In addition,

$$\begin{aligned}\alpha(x \oplus z)\beta([\mathrm{Ker}\, G\varphi]) &= \alpha(x)\alpha(z)\beta([\mathrm{Ker}\, G\varphi])\\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(x))\varepsilon Q(\mathrm{ind}_{\mathsf{T}}(z))\beta([\mathrm{Ker}\, G\varphi])\\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon\theta([\mathrm{Ker}\, G\varphi]),\\ &= (**)\end{aligned}$$

where the first = is due to α being exponential and the penultimate = is just by the definition of β.

Now, using the property that $i^*\overline{G} = G$, it follows that

$$[\mathrm{Ker}\, G\varphi] = [\mathrm{Ker}\, i^*\overline{G}\varphi] \overset{(1)}{=} [i^*\mathrm{Ker}\, \overline{G}\varphi] \overset{(2)}{=} \kappa[\mathrm{Ker}\, \overline{G}\varphi],$$

where (1) follows from i^* being an exact functor, and (2) from the definition of κ. We can now manipulate the expression $(**)$ further:

$$\begin{aligned}(**) &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon\theta(\kappa\big([\mathrm{Ker}\, \overline{G}\varphi]\big)\big)\\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon Q\big(\bar{\theta}([\mathrm{Ker}\, \overline{G}\varphi])\big)\\ &= \varepsilon(\mathrm{ind}_{\mathsf{T}}(x) + N)\varepsilon(\mathrm{ind}_{\mathsf{T}}(z) + N)\varepsilon\big(\bar{\theta}([\mathrm{Ker}\, \overline{G}\varphi]) + N\big)\\ &= (***),\end{aligned}$$

where the second equality is due to the commutativity of Diagram (4.4).

Comparing $(*)$ to $(***)$, we see that the required equality for Condition F is satisfied if

$$\bar{\theta}([\mathrm{Ker}\, \overline{G}\varphi]) = -(\mathrm{ind}_{\mathsf{T}}(C) + \mathrm{ind}_{\mathsf{T}}(\Sigma^{-1}C)). \tag{4.8}$$

Making use of the "rolling" property on our triangle in (4.7), we obtain the following sequence:

$$\Sigma^{-1}y \xrightarrow{\Sigma^{-1}\omega} \Sigma^{-1}z \xrightarrow{\Sigma^{-1}\psi} x \xrightarrow{\varphi} y \xrightarrow{\omega} z,$$

where any four consecutive terms form a triangle. Furthermore, since $\overline{G}$ is a homological functor, we may apply it to this sequence and produce the following long exact sequence in $\mathsf{fl}\,\mathsf{T}$:

$$\overline{G}\Sigma^{-1}y \xrightarrow{\overline{G}\Sigma^{-1}\omega} \overline{G}\Sigma^{-1}z \xrightarrow{\overline{G}\Sigma^{-1}\psi} \overline{G}x \xrightarrow{\overline{G}\varphi} \overline{G}y \xrightarrow{\overline{G}\omega} \overline{G}z. \tag{4.9}$$

This shows $\operatorname{Coker}\overline{G}\Sigma^{-1}\omega = \operatorname{Ker}\overline{G}\varphi$. Moreover, C is chosen such that $\overline{G}\Sigma^{-1}C = \operatorname{Coker}\overline{G}\Sigma^{-1}\omega$, and hence $\operatorname{Ker}\overline{G}\varphi = \overline{G}\Sigma^{-1}C$. We can hence compute as follows:

$$\begin{aligned}\bar{\theta}([\operatorname{Ker}\overline{G}\varphi]) &= \bar{\theta}([\overline{G}\Sigma^{-1}C]) \\ &= -(\operatorname{ind}_{\mathsf{T}}(\Sigma^{-1}C) + \operatorname{ind}_{\mathsf{T}}(\Sigma(\Sigma^{-1}C))) \\ &= -(\operatorname{ind}_{\mathsf{T}}(C) + \operatorname{ind}_{\mathsf{T}}(\Sigma^{-1}C)),\end{aligned}$$

where the second = is due to [15, lem. 2.10]. We can now see that Eq. (4.8) holds, and hence the lemma is proved. □

Remark 4.2 Through Lemma 4.1 and Theorem 3.2, we have managed to recover [15, thm. 2.11]. Indeed, [15, thm. 2.11] states that when

$$\Delta = \Sigma c \to b \to c$$

is an AR triangle in C such that $\overline{G}c$ and $\overline{G}(\Sigma c)$ have finite length in $\operatorname{Mod}\mathsf{T}$, the maps α and β from Eq. (4.5) satisfy the frieze-like condition given in [15, def. 1.4]. By Lemma 4.1, we know that α and β as defined in Eq. (4.5) satisfy Condition F. Theorem 3.2 proves that any α and β satisfying Condition F also satisfy the frieze-like condition for Δ (recovering [15, thm. 2.11]). Hence by [15, thm. 1.6], these α and β turn ρ into a generalised frieze.

5 The Multiplication Formula from [14]

In this section we demonstrate some of the technicalities behind the proof of the multiplication formula for π from Eq. (1.7), proved in [14, prop. 4.4]. This is done with a view of proving a similar formula for ρ in Sect. 6. Following the setup of [14, sec. 4], for this section we do not require a cluster tilting subcategory T, as the theory in [14] uses only the rigid subcategory R.

Let $m \in \operatorname{indec}\mathsf{C}$ and $r \in \operatorname{indec}\mathsf{R}$ be indecomposable objects such that $\operatorname{Ext}^1_{\mathsf{C}}(r,m)$ and $\operatorname{Ext}^1_{\mathsf{C}}(m,r)$ both have dimension one over $\mathbb{C}$. As in [14, rem. 4.2], this allows us to construct the following nonsplit triangles in C:

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \tag{5.1}$$

and

$$r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r, \tag{5.2}$$

with δ and ζ nonzero. Note that "rolling" the first triangle gives:

$$\Sigma^{-1}r \xrightarrow{-\Sigma^{-1}\delta} m \xrightarrow{\mu} a \xrightarrow{\gamma} r,$$

which is also a triangle in C. Applying the functor G to both the "rolled" triangle and the triangle in (5.2) gives the following exact sequences in $\operatorname{Mod}\mathsf{R}$, obtained in [14]:

$$G(\Sigma^{-1}r) \xrightarrow{-G(\Sigma^{-1}\delta)} Gm \xrightarrow{G\mu} Ga \longrightarrow 0$$

and

$$0 \longrightarrow Gb \xrightarrow{G\eta} Gm \xrightarrow{G\zeta} G(\Sigma r).$$

Remark 5.1 (1) The zeros arise in each exact sequence due to $G(r) = \mathrm{Hom}(-, \Sigma r)|_{\mathsf{R}}$ being the zero functor. Indeed, since R is rigid, evaluating $G(r)$ at any x in R will make the corresponding Hom-space zero.

(2) The exact sequences are in $\mathsf{Mod}\,\mathsf{R}$; that is, each term is a $\mathbb{C}$-linear contravariant functor $\mathsf{R} \to \mathsf{Vect}\,\mathbb{C}$.

Letting Gr denote the Grassmannian of submodules, we have morphisms of algebraic varietes,

$$\mathrm{Gr}(Ga) \xhookrightarrow{\ \xi\ } \mathrm{Gr}(Gm) \xhookleftarrow{\ \nu\ } \mathrm{Gr}(Gb),$$

$$P \longmapsto (G\mu)^{-1}(P),$$

$$(G\eta)(N) \longleftarrow\!\shortmid N.$$

It was proved in [14, Lemma 4.3] that if M in $\mathsf{Mod}\,\mathsf{R}$ is some subfunctor of Gm, then either $M \subseteq \mathrm{Im}\,G\eta$ or $\mathrm{Ker}\,G\mu \subseteq M$, but not both. This means that for $M \subseteq Gm$ we can find either a subfunctor $N \subseteq Gb$ such that $(G\eta)(N) = M$ or we can find $P \subseteq Ga$ such that $(G\mu)^{-1}(P) = M$. Hence, the subfunctor M is either of the form $(G\eta)(N)$ or $(G\mu)^{-1}(P)$.

It is hence clear that $\mathrm{Gr}(Gm)$ is isomorphic to the disjoint union of the images of ξ and ν. That is,

$$\mathrm{Gr}(Gm) \cong \mathrm{Gr}(Gb) \bigsqcup \mathrm{Gr}(Ga).$$

Remark 5.2 (1) We should note that for M in $\mathsf{Mod}\,\mathsf{R}$, the Grassmannian $\mathrm{Gr}(M)$ is an algebraic variety. Therefore, it makes sense to calculate the Euler characteristic of $\mathrm{Gr}(Gm)$, $\mathrm{Gr}(Ga)$ and $\mathrm{Gr}(Gb)$.

(2) In addition, we note that ξ and ν are both constructible maps, hence the images of ξ and ν form constructible subsets in $\mathrm{Gr}(Gm)$. See [17, Section 2.1] for the definitions of a constructible map and a constructible set.

The following statement then follows in [14]:

$$\chi(\mathrm{Gr}(Gm)) = \chi(\mathrm{Gr}(Gb)) + \chi(\mathrm{Gr}(Ga)), \tag{5.3}$$

where χ again denotes the Euler characteristic defined by cohomology with compact support (see [13, p. 93]). Using Remark 5.2, since the images of ξ and ν are constructible sets inside $\mathrm{Gr}(Gm)$, we know that χ is additive (see [13, p. 92, item (3)]), which gives the above equality in (5.3). That is

$$\pi(m) = \pi(a) + \pi(b).$$

6 Adaptation of the Multiplication Formula to [15]

This section builds on the material covered in the previous section and makes necessary adjustments and additions in order to obtain the multiplication formula for ρ, given in Theorem 6.2. Clearly, now that we are back working with ρ, we again require the setup of Sect. 2; that is, we need a cluster tilting subcategory T, with $\mathsf{R} \subseteq \mathsf{T}$.

As with π, we look to understand how to evaluate ρ for some $m \in \mathrm{indec}\,\mathsf{C}$. In the definition of ρ, we take a sum over $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$. In order to do this, we will require knowledge of the Grothendieck group $\mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$ and the K_0-classes of some of its key elements.

Firstly, we know

$$[\nu N] = [N].$$

Indeed, by definition, $\nu N = (G\eta)(N)$, and since $G\eta$ is injective, $(G\eta)(N)$ and N have the same composition series. Hence, the above equality is true.

To find $[\xi P]$, we first note that by definition, $\xi P = (G\mu)^{-1}P$, and therefore, $[\xi P] = [(G\mu)^{-1}P]$. A consequence of the Second Isomorphism Theorem is that a composition series of $(G\mu)^{-1}P$ can be obtained by concatenating composition series of P and of $\operatorname{Ker} G\mu$. That is, $[(G\mu)^{-1}(P)] = [P] + [\operatorname{Ker} G\mu]$. So, the K_0-classes are:

$$[\nu N] = [N] \text{ and } [\xi P] = [P] + [\operatorname{Ker} G\mu].$$

Now that we have some useful information about the K_0-classes, we can take a more in depth look at ρ. Consider r in R, and let us calculate $\rho(r)$:

$$\begin{aligned}\rho(r) &= \alpha(r)\sum_e \chi\Big(\mathrm{Gr}_e(G(r))\Big)\beta(e)\\ &= \alpha(r)\sum_e \chi(\mathrm{Gr}_e(0))\beta(e)\\ &= \alpha(r)\beta(0)\\ &= \alpha(r).\end{aligned}$$

In the above calculation, the third = is due to $\chi(\mathrm{Gr}_e(0))$ being zero for all nonzero $e \in \mathrm{K}_0(\mathsf{fl}\,\mathsf{R})$ and one when $e = 0$. The last = is since β is exponential.

Now consider $\rho(r)\rho(m)$ for $m \in \operatorname{indec}\mathsf{C}$:

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_e \chi(\mathrm{Gr}_e(Gm))\beta(e)\\ &= \alpha(r)\alpha(m)\Big(\sum_e \chi(\operatorname{Im}\xi \cap \mathrm{Gr}_e(Gm)) + \chi(\operatorname{Im}\nu \cap \mathrm{Gr}_e(Gm))\Big)\beta(e),\end{aligned}$$

where the second equality arises from $\mathrm{Gr}(Gm)$ being the disjoint union of the images of ξ and ν. We now make an important remark about the two intersections in the second equality above.

Remark 6.1 (1) The first intersection is given by the image of ξ when applied to $\mathrm{Gr}_e(Ga)$. Indeed,

$$\begin{aligned}\operatorname{Im}\xi \cap \mathrm{Gr}_e(Gm) &= \{\xi P \mid [\xi P] = e\}\\ &= \{\xi P \mid [P] = e - [\operatorname{Ker} G\mu]\}\\ &= \xi(\mathrm{Gr}_{e-[\operatorname{Ker} G\mu]}(Ga)).\end{aligned}$$

Here we used the fact that $[\xi P] = [P] + [\operatorname{Ker} G\mu]$.

(2) The second intersection can be obtained in a similar way:

$$\begin{aligned}\operatorname{Im}\nu \cap \mathrm{Gr}_e(Gm) &= \{\nu N \mid [\nu N] = e\}\\ &= \{\nu N \mid [N] = e\}\\ &= \nu(\mathrm{Gr}_e(Gb))\end{aligned}$$

Using this remark, we can continue to calculate $\rho(r)\rho(m)$, obtaining:

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_e \Big(\chi(\xi(\mathrm{Gr}_{e-[\mathrm{Ker}\,G\mu]}(Ga)) + \chi(\nu(\mathrm{Gr}_e(Gb)))\Big)\beta(e) \\ &= \alpha(r)\alpha(m)\sum_e \Big(\chi(\mathrm{Gr}_{e-[\mathrm{Ker}\,G\mu]}(Ga)) + \chi(\mathrm{Gr}_e(Gb))\Big)\beta(e). \qquad (6.1)\end{aligned}$$

We can discard ξ and ν in the final expression since they are both embeddings.

Theorem 6.2 *Let $m \in \operatorname{indec} \mathsf{C}$ and $r \in \operatorname{indec} \mathsf{R}$ such that $\mathrm{Ext}^1_{\mathsf{C}}(r,m)$ and $\mathrm{Ext}^1_{\mathsf{C}}(m,r)$ both have dimension one over $\mathbb{C}$. Then, there are nonsplit triangles*

$$m \xrightarrow{\mu} a \xrightarrow{\gamma} r \xrightarrow{\delta} \Sigma m \qquad \text{and} \qquad r \xrightarrow{\sigma} b \xrightarrow{\eta} m \xrightarrow{\zeta} \Sigma r,$$

with δ and ζ nonzero. Let Gm have finite length in $\mathsf{Mod}\,\mathsf{R}$, then

$$\rho(r)\rho(m) = \rho(a) + \rho(b).$$

Proof We first note that since Gm has finite length, then so do Ga and Gb. This follows immediately from [14, rem. 4.2].

Now, by making the substitution $f = e - [\mathrm{Ker}\,G\mu]$ in Eq. (6.1), observe that

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f + [\mathrm{Ker}\,G\mu]) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e) \\ &\overset{(a)}{=} \alpha(r)\alpha(m)\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f)\beta([\mathrm{Ker}\,G\mu]) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e) \\ &\overset{(b)}{=} \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\mu])\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f) \\ &\quad + \alpha(r)\alpha(m)\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e). \qquad (6.2)\end{aligned}$$

Here (a) is due to β being exponential (see [15, setup 1.1]) and (b) is due to $\beta([\mathrm{Ker}\,G\mu])$ being a constant.

Now, consider $\mathrm{Ker}\,G\sigma$. Since $G(r) = 0$, then $\mathrm{Ker}\,G\sigma = 0$, and clearly $[\mathrm{Ker}\,G\sigma] = 0$. The map β is exponential, and therefore $\beta([\mathrm{Ker}\,G\sigma]) = \beta(0) = 1$. We can insert this in to Eq. (6.2) and see that:

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\mu])\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f) \\ &\quad + \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\sigma])\sum_e \chi\big(\mathrm{Gr}_e(Gb)\big)\beta(e). \qquad (6.3)\end{aligned}$$

Applying Condition F to our two triangles in the theorem, whilst remembering that α is exponential, gives

$$\begin{aligned}\alpha(a) &= \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\mu]) \\ \alpha(b) &= \alpha(r)\alpha(m)\beta([\mathrm{Ker}\,G\sigma]).\end{aligned}$$

Returning these equalities into Eq. (6.3), the expression for $\rho(r)\rho(m)$ becomes:

$$\begin{aligned}\rho(r)\rho(m) &= \alpha(a)\sum_f \chi\big(\mathrm{Gr}_f(Ga)\big)\beta(f) + \alpha(b)\sum_e \big(\mathrm{Gr}_e(Gb)\big)\beta(e)\\ &= \rho(a)+\rho(b).\end{aligned}$$

□

7 Example for $\mathsf{C}(A_5)$

In this section, we will demonstrate the multiplication formula for ρ in Theorem 6.2 by recomputing a vertex in the AR quiver in Fig. 3. We first give some brief background on the polygon model for $\mathsf{C} = \mathsf{C}(A_n)$, the cluster category of Dynkin type A_n. Here, $n \geq 2$ is an integer. By [7], the indecomposables of C can be identified with the diagonals of a regular $(n+3)$-gon P with the set of vertices $\{0, \ldots, n+2\}$. By [14, thm. 5.4], the indecomposables of the rigid subcategory $\mathsf{R} \subseteq \mathsf{T}$ give a polygon dissection of P, and by [4] the cluster tilting subcategory T gives a full triangulation of the $(n+3)$-gon. Indeed, recall that there is a full subcategory S, which is closed under direct sums and summands, such that

$$\operatorname{indec}\mathsf{T} = \operatorname{indec}\mathsf{R} \cup \operatorname{indec}\mathsf{S}.$$

Hence, the indecomposables in S correspond to a triangulation of each of the cells of P given by the polygon dissection from $\operatorname{indec}\mathsf{R}$. We should note in addition that each edge of the $(n+3)$-gon is identified with the zero object inside C.

This model also comes with the convenient property that for two indecomposables a and b in C

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(a,b) = \begin{cases} 1, & \text{if } a \text{ and } b \text{ cross}\\ 0, & \text{otherwise.}\end{cases}$$

It is known by Theorem 6.2 that for $m \in \operatorname{indec}\mathsf{C}$ and $r \in \operatorname{indec}\mathsf{R}$ such that $\dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(m,r) = \dim_{\mathbb{C}}\mathrm{Ext}^1_{\mathsf{C}}(r,m) = 1$, then

$$\rho(r)\rho(m) = \rho(a)+\rho(b),$$

where a and b are the middle terms of the nonsplit extensions in (5.1) and (5.2). In the case of $\mathsf{C} = \mathsf{C}(A_n)$, a and b can be obtained as seen in the polygon in Fig. 4, where we have $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$. See [15, sec. 5] for details.

Now, to compute our example, we refer to the setup of [15, sec. 3]; that is, we set $\mathsf{C} = \mathsf{C}(A_5)$, the cluster category of Dynkin type A_5. Thus, the indecomposables of C can be identified with the diagonals on a regular 8-gon. As in [15] we will denote by $\{a, b\}$ the indecomposable corresponding to the diagonal connecting the vertices a and b. We use the same polygon triangulation as in [15, sec. 3]; that is, $\operatorname{indec}\mathsf{R}$ corresponds to the dotted diagonals in Fig. 5 and $\operatorname{indec}\mathsf{S}$ corresponds to the dashed diagonals. Hence, R contains the following indecomposable objects :

$$\{2,5\}\ ,\ \{2,7\},$$

whilst the indecomposables in S are:

$$\{1,7\}\ ,\ \{2,4\}\ ,\ \{5,7\}.$$

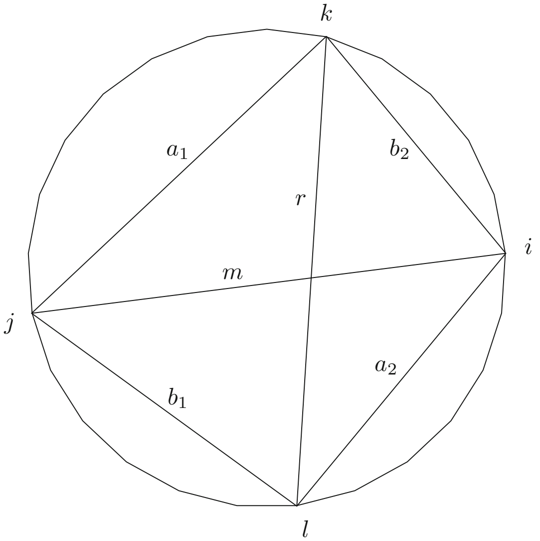

Fig. 4 There are nonsplit extensions $m \to a_1 \oplus a_2 \to r$ and $r \to b_1 \oplus b_2 \to m$

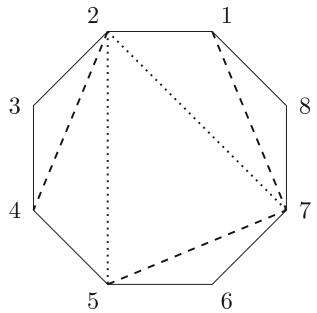

Fig. 5 Dotted diagonals correspond to indecomposables in indec R and dashed diagonals correspond to indecomposables in indec S

These indecomposables in S fit in the following exchange triangles:

$$\begin{array}{ccc}
\{1,7\} \longrightarrow \{2,7\} \longrightarrow \{2,8\} & \qquad & \{2,8\} \longrightarrow 0 \longrightarrow \{1,7\} \\
\{2,4\} \longrightarrow \{2,5\} \longrightarrow \{3,5\} & \qquad & \{3,5\} \longrightarrow 0 \longrightarrow \{2,4\} \\
\{5,7\} \longrightarrow \{2,5\} \longrightarrow \{2,6\} & \qquad & \{2,6\} \longrightarrow \{2,7\} \longrightarrow \{5,7\}.
\end{array}$$

Then, applying the definition of N from (4.3), it is easily seen that

$$N = \big\langle [2,5], [2,7] \big\rangle,$$

Here, we denote by $[a,b]$ the $\mathrm{K}_0^{\mathrm{split}}$-class of the indecomposable $\{a,b\}$. We also have

$$\mathrm{K}_0^{\mathrm{split}}(\mathsf{T})/N = \Big\langle [1,7]+N,\ [2,4]+N,\ [5,7]+N \Big\rangle,$$

and let the exponential map $\varepsilon : \mathrm{K}_0^{\text{split}}(\mathsf{T})/N \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}, z^{\pm 1}]$ be given by:

$$\varepsilon([1,7]+N) = u\ ,\ \varepsilon([2,4]+N) = v\ ,\ \varepsilon([5,7]+N) = z. \tag{7.1}$$

We will now demonstrate how to calculate $\rho(\{4,6\})$ using an alternative method to that in [15, ex. 3.5]. We will compute it by applying the multiplication formula for ρ in Theorem 6.2. Since $\dim_{\mathbb{C}}\mathrm{Ext}^1(\{4,6\},\{2,5\}) = \dim_{\mathbb{C}}\mathrm{Ext}^1(\{2,5\},\{4,6\}) = 1$, we may set $r = \{2,5\}$, and using Fig. 4, we know that $\{4,6\}$ sits in the following nonsplit extensions:

$$\{4,6\} \to \{2,4\} \to \{2,5\}\ ,\ \{2,5\} \to \{2,6\} \to \{4,6\}.$$

Applying Theorem 6.2, we get the following equality:

$$\rho(\{2,5\})\rho(\{4,6\}) = \rho(\{2,4\}) + \rho(\{2,6\}). \tag{7.2}$$

Due to the fact that the diagonals corresponding to $\{2,5\}$, $\{2,4\}$ and $\{2,6\}$ do not cross any diagonals in $\operatorname{indec}\mathsf{R}$, it is immediate that

$$G(\{2,5\}) = G(\{2,4\}) = G(\{2,6\}) = 0.$$

Hence, by the definition of ρ, Eq. (7.2) becomes:

$$\alpha(\{2,5\})\rho(\{4,6\}) = \alpha(\{2,4\}) + \alpha(\{2,6\}). \tag{7.3}$$

In order to calculate α of each of the indecomposables $\{2,5\}$, $\{2,4\}$ and $\{2,6\}$, we first calculate their indices with respect to T.

$\{2,5\}$ sits in the following triangle:

$$0 \to \{2,5\} \to \{2,5\},$$

and since $\{2,5\} \in \operatorname{indec}\mathsf{T}$, we see that:

$$\operatorname{ind}_{\mathsf{T}}(\{2,5\}) = [2,5]. \tag{7.4}$$

By the same logic,

$$\operatorname{ind}_{\mathsf{T}}(\{2,4\}) = [2,4]. \tag{7.5}$$

We note that one of the exchange triangles for $\{2,6\}$ is:

$$\{5,7\} \to \{2,5\} \to \{2,6\},$$

and hence

$$\operatorname{ind}_{\mathsf{T}}(\{2,6\}) = [2,5] - [5,7]. \tag{7.6}$$

Since $[2,5] \in N$, using the definition of α from [15, def. 2.8], we see that:

$$\begin{aligned}\alpha(\{2,5\}) &= \varepsilon Q(\operatorname{ind}_{\mathsf{T}}(\{2,5\})\\ &= \varepsilon Q([2,5])\\ &= \varepsilon([2,5]+N)\\ &= \varepsilon(0)\\ &= 1,\end{aligned}$$

and hence, the right hand side of Eq. (7.3) becomes $\alpha(\{2,5\})\rho(\{4,6\}) = \rho(\{4,6\})$. Substituting back into Eq. (7.3), we obtain:

$$\begin{aligned}\rho(\{4,6\}) &= \alpha(\{2,4\}) + \alpha(\{2,6\}) \\ &= \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,4\})) + \varepsilon Q(\mathrm{ind}_{\mathsf{T}}(\{2,6\})) \\ &\overset{(1)}{=} \varepsilon Q([2,4]) + \varepsilon Q([2,5]-[5,7]) \\ &= \varepsilon([2,4]+N) + \varepsilon(-[5,7]+N) \\ &\overset{(2)}{=} v + z^{-1} \\ &= \frac{1+vz}{z},\end{aligned}$$

where (1) is by substituting the values from Eqs. (7.5) and (7.6), and (2) is due to the definition of ε from Eq. (7.1). We notice here that this is indeed the same result for $\rho(\{4,6\})$ obtained in [15, ex. 3.5].

Similar computations for the other indecomposables in $\mathsf{C}(A_5)$ will produce the generalised frieze as drawn in Fig. 3 in the introduction.

Remark 7.1 In general, the formula from Theorem 6.2 can be applied iteratively to calculate $\rho(m)$. Indeed,

$$\rho(r)\rho(m) = \rho(a) + \rho(b)$$

is an iterative formula on m, and hence, calculating ρ of each indecomposable in C can be reduced to calculating ρ of the indecomposables in C whose corresponding diagonals in the $(n+3)$-gon do not cross any of the diagonals in R. Namely, it is clear from Fig. 4 that each of a_1, a_2, b_1 and b_2 sit inside "smaller" polygons than m. Here, the smaller polygons are those obtained from r dissecting the (n+3)-gon. Since R consists of only non-crossing diagonals, the remaining diagonals in R sit inside these smaller polygons. Reapplying Theorem 6.2 to each of a_1, a_2, b_1 and b_2 will again create a series of even smaller polygons, containing a new a or b. After repeated iterations, this process will eventually terminate at the stage when the new a or b does not cross any of the diagonals in R.

Now, in the case when a diagonal, say m', does not cross a diagonal in R, calculating $\rho(m')$ is acheived by calculating $\alpha(m')$. This is clear since $G(m') = 0$. Computing $\alpha(m')$ is done by calculating the index of m', and then applying the maps Q and ε. Hence, finding $\rho(m)$ for each $m \in \mathrm{indec}\,\mathsf{C}$ can be reduced by Theorem 6.2 to computing the index of each of the indecomposables in C whose corresponding diagonals do not cross any diagonals in R.

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References

1. Assem, I., Dupont, G.: Friezes and a construction of the Euclidean cluster variables. J. Pure Appl. Algebra **215**, 2322–2340 (2011). https://doi.org/10.1016/j.jpaa.2012.12.013
2. Auslander, M.: Representation theory of Artin algebras II. Commun. Algebra **1**, 269–310 (1974)

3. Buan, A.B., Iyama, O., Reiten, I., Scott, J.: Cluster structure for 2-Calabi–Yau categories and unipotent groups. Compos. Math. **145**, 1035–1079 (2009)
4. Buan, A.B., Marsh, R., Reineke, M., Reiten, I., Todorov, G.: Tilting theory and cluster combinatorics. Adv. Math. **204**, 572–618 (2006)
5. Brüstle, T., Zhang, J.: On the cluster category of a marked surface without punctures. Algebra Number Theory **5**(4), 529–566 (2011)
6. Caldero, P., Chapoton, F.: Cluster algebras as Hall algebras of quiver representations. Comment. Math. Helv. **81**, 595–616 (2006). https://doi.org/10.4171/CMH/65
7. Caldero, P., Chapoton, F., Schiffler, R.: Quivers with relations arising from clusters (A_n case). Trans. Am. Math. Soc. **358**, 1347–1364 (2006)
8. Caldero, P., Keller, B.: From triangulated categories to cluster algebras. Invent. Math. **172**, 169–211 (2008)
9. Conway, J.H., Coxeter, H.S.M.: Triangulated polygons and frieze patterns. Math. Gaz. **57**(400), 87–94 (1973)
10. Conway, J.H., Coxeter, H.S.M.: Triangulated polygons and frieze patterns. Math. Gaz. **57**(401), 175–183 (1973)
11. Dehy, R., Keller, B.: On the combinatorics of rigid objects in 2-Calabi–Yau categories. Int. Math. Res. Not. (11), (2008)
12. Domínguez, S., Geiss, C.: A Caldero–Chapoton formula for generalized cluster categories. J. Algebra **399**, 887–893 (2014). https://doi.org/10.1016/j.jalgebra.2013.10.018
13. Fulton, W.: Introduction to Toric Varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton (1993)
14. Holm, T., Jørgensen, P.: Generalized friezes and a modified Caldero–Chapoton map depending on a rigid object. Nagoya Math. J. **218**, 101–124 (2015)
15. Holm, T., Jørgensen, P.: Generalized friezes and a modified Caldero–Chapoton map depending on a rigid object II. Bull. Sci. Math. **140**, 112–131 (2016)
16. Marsh, R., Reineke, M., Zelevinsky, A.: Generalized associahedra via quiver representations. Trans. Am. Math. Soc. **355**(10), 4171–4186 (2003)
17. Palu, Y.: Cluster characters, II: a multiplication formula. Proc. Lond. Math. Soc. **3**(104), 57–78 (2012)